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Geometry of obstructed families of curves

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Abstract

We study certain equisingular families of curves with quasihomogeneous singularity of minimal obstructness (i.e. $h^1 = 1$). We show that our families always have expected codimension. Moreover they are either non-reduced with smooth reduction or decompose into two smooth components of expected codimension that intersect non-transversally or are reduced irreducible non-smooth varieties which have smooth singular locus with sectional singularity of type A_1. On the other hand there is an example of an equisingular family of curves with multiple quasihomogeneous singularities of minimal obstructness which is smooth but has wrong codimension. We use algorithms of computer algebra as a technical tool.
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1. Introduction

The study of equisingular families of algebraic curves with given invariants and given set of singularities is an old, but still attractive and widely open problem. Already at the beginning of the 20th century, the foundation was laid in the works of Plücker, Severi, Segre and Zariski. Later the theory of equisingular families was a focus of numerous studies by algebraic geometers and it has found important applications in singularity theory, in the topology of complex algebraic curves and surfaces, and in real algebraic geometry.

We consider families of algebraic curves C on a smooth projective surface Σ over the complex field $\mathbb{C}$. Let D be a divisor on Σ and $V = V_{|D|}(S_1, \ldots, S_r)$ be the variety of curves in the linear system $|D|$ having r singular points of (topological or analytic) types $S_1, \ldots, S_r$ as the only singularities. One knows that $V_{|D|}(S_1, \ldots, S_r)$ can be identified with a (locally closed) subscheme ("*equisingular stratum*") in the Hilbert scheme of curves on Σ. The main questions concerning this space are:

- *Existence problem:* Is $V_{|D|}(S_1, \ldots, S_r)$ non-empty, that is, does there exist a curve $F \in |D|$ with the given collection of singularities?
- *Smoothness problem:* If $V_{|D|}(S_1, \ldots, S_r)$ is non-empty, is it smooth and of "expected" dimension (expressible via local invariants of the singularities)?
- *Irreducibility problem:* Is $V_{|D|}(S_1, \ldots, S_r)$ irreducible?

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No complete solution to the above problems is known except basically for the case of plane nodal curves. The methods of deformation theory and cohomology vanishing theory have led to sufficient conditions for the positive answers to the problems in question. However, there is a large gap between sufficient and necessary conditions; furthermore, there are examples of equisingular families which are not smooth or are reducible (cf. [13]).

The well-known classical result is that the variety V is T-smooth at $C \subset V$ if and only if

$$h^1(\mathcal{J}_{Z^{es}(C)/\Sigma}(C)) = 0, \quad \text{respectively } h^1(\mathcal{J}_{Z^{ea}(C)/\Sigma}(C)) = 0.$$

We want to consider the equisingular families of curves for which the condition is not satisfied, that is where T-smoothness fails. Such equisingular families are called *obstructed*.

By the moment, obstructed families of curves have not been studied systematically and are represented by a few series of examples (cf. Examples 1.1 and 1.2, [6,9–11,13]).

A classical example of Segre [8,12] gives a series of irreducible plane curves such that the corresponding germs of the equisingular strata are non-T-smooth:

Example 1.1. Let F_{2m} and G_{3m}, respectively, be two generic homogeneous polynomials in three variables of degrees $2m$ and $3m$. Then the curve $C_{6m} \subset \mathbb{P}^2$ of degree $d = 6m$ given by

$$(F_{2m}(x, y, z))^3 + (G_{3m}(x, y, z))^2 = 0$$

is irreducible and has precisely $6m^2$ cusps as the only singularities. Moreover, the dimension of the family $V' \subset V_{6m}(6m^2 \cdot A_2)$ of such curves is

$$\frac{2m(2m+3)}{2} + \frac{3m(3m+3)}{2} + 1 = \frac{d(d+3)}{2} - 12m^2 + \frac{(m-1)(m-2)}{2}.$$

It follows that for $m \geq 3$ the variety $V_{6m}(6m^2 \cdot A_2)$ contains a component which has a bigger dimension than the expected one. In particular, this component is non-T-smooth.

On the other hand, $h^1(\mathcal{J}_{Z^{ea}(C_{6m})/\mathbb{P}^2}(6m)) = \frac{(m-1)(m-2)}{2}$, whence we can conclude that $V_{6m}(6m^2 \cdot A_2)$ is smooth at C_{6m}.

One of the interesting recent examples is due to du Plessis and Wall [1]:

Example 1.2. (a) For any $d \geq 5$ the curve $C \subset \mathbb{P}^2$ given by the equation $(x_1^d + x_2^5 x_0^{d-5} + x_2^d = 0)$ has unique singular point $z = (0:0:1)$ with Tjurina number $\tau(C, z) = 4d - 4$, and satisfies

$$h^1(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)) > 0.$$

(b) Denote by S the analytic type of plane curve singularity (C, z). If $d \geq 10$ then the family $V_d(S)$ is singular at C.

We make the next step in the study of geometry of obstructed families of singular curves and focus on the minimal obstructness case, i.e. $h^1(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)) = 1$. We study the series of examples of du Plessis and Wall and prove the following result:

Theorem 1.1. *Let C be a projective plane curve, given in local coordinates $x = \frac{x1}{x0}$, $y = \frac{x2}{x0}$ by the equation $x^d + y^5 = 0$.*

For any $d > 5$ the germ $V_{d,C}(S)$ at C of the equianalytic family of plane curves $V_d(S)$, where S is the analytic type of the plane curve singularity (C, z), is non-T-smooth and $h^1(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)) = 1$. Furthermore:

- (i) *For $d = 6$ the germ $V_{6,C}(S)$ is non-reduced. It is a double $[V_{6,C}(S)]_{red}$, and $[V_{6,C}(S)]_{red}$ is smooth of expected codimension.*
- (ii) *For $d = 7$ the germ $V_{7,C}(S)$ is reducible and decomposes into two smooth components of expected codimension that intersect non-transversally with multiplicity one. The intersection locus is smooth. Moreover, the sectional singularity is of type A_1.*
- (iii) *For $d \geq 8$ the germ $V_{d,C}(S)$ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus with sectional singularity of type A_1.*

Furthermore, we prove the following generalization:

Theorem 1.2. *Let C be a projective plane curve, given in local coordinates $x = \frac{x1}{x0}$, $y = \frac{x2}{x0}$ by the equation $x^k + y^l = 0$. Suppose for convenience $k \geq l$. Let $d = k + l - 5$ and let $V_{d,C}(S)$ be the germ at C of the equianalytic family of plane curves $V_d(S)$, where S is the analytic type of the plane curve singularity (C, z). For any $k, l \geq 5$ such that $d > 5$, $V_{d,C}(S)$ is non-T-smooth and $h^1(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)) = 1$. Furthermore:*

(i) *If $d = 6$ (i.e. $l = 5, k = 6$), the germ $V_{6,C}(S)$ is non-reduced. It is a double $[V_{6,C}(S)]_{red}$, and $[V_{6,C}(S)]_{red}$ is smooth of expected codimension.*
(ii) *If $d = 7$ (i.e. $l = 5, k = 7$ or $k = l = 6$), the germ $V_{7,C}(S)$ is reducible and decomposes into two smooth components of expected codimension that intersect non-transversally with multiplicity one. The intersection locus is smooth. Moreover, the sectional singularity is of type A_1.*
(iii) *If $d \geq 8$, the germ $V_{d,C}(S)$ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus with sectional singularity of type A_1.*

Summarizing the conclusions of the theorems we formulate the following

Conjecture. *Let C be a projective curve with a singular point z and $h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) = 1$. Consider the germ $V_{d,C}(S)$ at C of the equianalytic family of plane curves $V_d(S)$ (where S is the analytic singularity type of (C, z)). Then one of the following holds:*

(i) *$V_{d,C}(S)$ is non-reduced with a smooth reduction of expected codimension and $h^1(\mathcal{J}_{Z^{ea}(C',z)/\mathbb{P}^2}(d)) = 1$ for any $C' \in V_{d,C}(S)$.*
(ii) *$V_{d,C}(S)$ is singular with a smooth singular locus and has expected codimension. $h^1(\mathcal{J}_{Z^{ea}(C',z)/\mathbb{P}^2}(d)) = 1$ for C' in the singular locus and $h^1(\mathcal{J}_{Z^{ea}(C',z)/\mathbb{P}^2}(d)) = 0$ outside it. In this case the germ has order two along the singular locus.*
(iii) *$V_{d,C}(S)$ is smooth of codimension one less then the expected one and $h^1(\mathcal{J}_{Z^{ea}(C',z)/\mathbb{P}^2}(d)) = 1$ for any $C' \in V_{d,C}(S)$.*

The methods that we use are the technique of cohomologies of ideal sheaves of zero-dimensional schemes associated with analytic types of singularities, methods for their calculation, and H^1-vanishing theorems. We also use the algorithms of computer algebra (cf. [5]) as a technical tool in the proof of the theorems.

2. Preliminaries

2.1. Singularities of algebraic curves

In this section we define the different types of reduced plane curve singularities which we consider throughout this article. For more detailed description of the notions appearing in this and the next subsections cf. [4].

2.1.1. Analytic types

We introduce the concept of analytic types in the more general context of germs of isolated hypersurface singularities $(F, z) \in (\Sigma, z)$, where Σ is an n-dimensional smooth projective variety.

Definition 2.1. Two germs $(F, z) \in (\Sigma, z)$ and $(G, w) \in (\Sigma, w)$ of isolated hypersurface singularities (or any of their defining power series) are said to be *analytically equivalent* if there exists a local analytic isomorphism $(\Sigma, z) \to (\Sigma, w)$ mapping (F, z) to (G, w). The corresponding equivalence classes are called *analytic types*.

Remark 2.2. Let S be an analytic type of reduced hypersurface singularity represented by $(C, z) \in (\Sigma, z)$; $f \in \mathbb{C}\{x_1, \dots, x_n\}$ is a local equation for (C, z). Define $j(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$.

We introduce the following invariants: the *Milnor number*

$$\mu(S) = \mu(C, z) := \dim_{\mathbb{C}} \mathbb{C}\{x\}/j(f)$$

and the *Tjurina number*

$$\tau^{ea}(S) := \tau(S) = \tau(C, z) := \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle f, j(f) \rangle$$

for analytic types S.

Let Σ be a smooth projective surface and $C \subset \Sigma$ a reduced curve with singular locus $\mathrm{Sing}(C) = \{z_1, \dots, z_r\}$ and D be a divisor on Σ.

Definition 2.3. $V_{|D|}(S_1, \dots, S_r) \subset |D|$ and $V^{\mathrm{irr}}_{|D|}(S_1, \dots, S_r) \subset |D|$, respectively, denote the locally closed subspaces of reduced and irreducible curves $C' \subset \Sigma$ having singularities of the types $S_1, \dots, S_r$.

For $\Sigma = \mathbb{P}^2$ we consider the curves in the linear system $|dH|$, where H is a hyperplane section, and denote the corresponding varieties by $V_d = V_d(S_1, \dots, S_r)$.

2.1.2. Quasihomogeneous singularities

Definition 2.4. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \subset \mathbb{C}[x_1, \dots, x_n]$

(i) The polynomial f is called *quasihomogeneous of type*

$$(w; d) = (w_1, \dots, w_n; d)$$

if w_i, d are positive integers satisfying

$$\langle w, \alpha \rangle = w_1\alpha_1 + \cdots w_n\alpha_n = d$$

for each $\alpha \in \mathbb{N}^n$ with $a_\alpha \neq 0$.

(ii) An isolated hypersurface singularity $(X, x) \subset (\mathbb{C}^n, x)$ is called *quasihomogeneous* if there exists a quasihomogeneous polynomial $f \in C[x_1, \dots, x_n]$ such that $\mathcal{O}_{C,x} \cong \mathbb{C}\{x\}/\langle f \rangle$.

A quasihomogeneous polynomial f of type $(w; d)$ obviously satisfies the relation

$$d \cdot f = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i},$$

that implies that f is contained in $j(f)$ and, hence, for quasihomogeneous isolated hypersurface singularities $\mu = \tau$. It is a theorem of Saito [7] that for an isolated singularity the converse also holds. So, let $(X, x) \subset (\mathbb{C}^n, x)$ be an isolated hypersurface singularity and let $f \subset \mathbb{C}\{x_1, \dots, x_n\}$ be an any local equation for (X, x); then

$$(X, x) \quad \text{is quasihomogeneous} \quad \Longleftrightarrow \quad f \in j(f) \quad \Longleftrightarrow \quad \mu(X, x) = \tau(X, x).$$

2.2. Zero-dimensional schemes associated with singularities

Let Σ be a smooth projective surface and $C \subset \Sigma$ a reduced curve. For any $z \in \mathrm{Sing}(C)$ let f be the local equation of (C, z) in (Σ, z). Let $j(f) = \langle \frac{\partial f}{\partial x_i} \rangle$ be the jacobian of f and $I^{ea}(C, z) := \langle f, j(f) \rangle \subset \mathcal{O}_{\Sigma,z}$ be the Tjurina ideal.

Let $Z' \subset \Sigma$ be a zero-dimensional scheme. It is defined by a finite set of points $\mathcal{S}' = \mathrm{Supp}(Z')$ and ideals $I'(z) \subset \mathcal{O}_{\Sigma,z}$ for all $z \in \mathcal{S}'$. Let V' be the subfamily of reduced curves in $|C|$ such that for any $z \in \mathcal{S}'$ the deformations of (C, z) over $T_\varepsilon = \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ defined by V' are given by $f + \varepsilon g$ where $g \in I'(z)$.

Example 2.5. For $\mathcal{S}' = \mathrm{Sing}(C) = \{z_1, \dots, z_r\}$ and $I'(z_i) = I^{ea}(C, z_i)$ we have $V' = V_{|C|}(S_1, \dots, S_r)$, where S_i are the analytic types of singularities (C, z_i). The zero-dimensional scheme Z' that we obtain in this case is denoted as $Z^{ea}(C)$.

Example 2.6. For $\mathcal{S}' = \{z\} \subset \mathrm{Sing}(C)$ and $I'(z) = I^{ea}(C, z)$ we have $V' = V_{|C|}(S)$, where S is the analytic type of singularities (C, z). We denote the zero-dimensional scheme Z' that we obtain in this case by $Z^{ea}(C, z)$.

Example 2.7. More generally, if $\mathcal{S}'' = \{z_1, \dots, z_k\} \subset \mathrm{Sing}(C) \cap \mathcal{S}'$, and $I'(z) \subset I^{ea}(C, z)$ for all $z \in \mathcal{S}''$, then $V' \subset V_{|C|}(S_1, \dots, S_k)$ where S_i are the analytic types of singularities (C, z_i).

The importance of the schemes Z' comes from the fact that the cohomology groups $H^i(\mathcal{J}_{Z'/\Sigma}(C))$ have a precise geometric meaning for the space V', as we can see in the following proposition which summarizes the classically known facts. As in this article we work only with plane projective curves we will formulate it for $\Sigma = \mathbb{P}^2$.

Proposition 2.8. *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree d.*
(a) $H^0(\mathcal{J}_{Z'/\mathbb{P}^2}(d))/H^0(\mathcal{O}_{\mathbb{P}^2})$ *is isomorphic to the Zariski tangent space of V' at C.*
(b) $h^0(\mathcal{J}_{Z'/\mathbb{P}^2}(d)) - h^1(\mathcal{J}_{Z'/\mathbb{P}^2}(d)) - 1 \leq \dim(V', C) \leq h^0(\mathcal{J}_{Z'(C)/\mathbb{P}^2}(d)) - 1.$
(c) $H^1(\mathcal{J}_{Z'/\mathbb{P}^2}(d)) = 0$ *if and only if V' is T-smooth at C, that is, smooth of the expected dimension* $\frac{d(d+3)}{2} - \deg Z'$.

For the proof of this proposition cf. [2,3].

2.3. Notions of computer algebra

In our computations we want to use methods of computer algebra. Here we introduce the basic notions, that will be widely used in our proofs. A more detailed description of these notions can be found also in [5].

2.3.1. Monomial orderings

Definition 2.9. A *monomial ordering* is a total (or linear) ordering $>$ on the set of monomials $\mathrm{Mon}_n = \{x^\alpha \mid \alpha \in \mathbb{N}^n\}$ in n variables satisfying

$$x^\alpha > x^\beta \Rightarrow x^\gamma x^\alpha > x^\gamma x^\beta$$

for all $\alpha, \beta, \gamma \in \mathbb{N}^n$. We say also that $>$ is a monomial ordering on $A[x_1, \ldots, x_n]$, where A is any ring, meaning that $>$ is a monomial ordering on Mon_n.

We identify Mon_n with $\mathbb{N}^n$, and then a monomial ordering is a total ordering on $\mathbb{N}^n$, which is compatible with the semigroup structure on $\mathbb{N}^n$ given by addition. From a practical point of view, a monomial ordering $>$ allows us to write a polynomial $f \in K[x]$ in a unique ordered way as

$$f = a_\alpha x^\alpha + a_\beta x^\beta + \cdots + a_\gamma x^\gamma,$$

with $x^\alpha > x^\beta > \cdots > x^\gamma$, where no coefficient is zero. The most important distinction is between global and local orderings.

Definition 2.10. Let $>$ be a monomial ordering on $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$;
(1) $>$ is called a *global* ordering if $x^\alpha > 1$ for all $\alpha \neq (0, \ldots, 0)$,
(2) $>$ is called a *local* ordering if $x^\alpha < 1$ for all $\alpha \neq (0, \ldots, 0)$

Important examples of monomial orderings are:

Example 2.11 (*Monomial Orderings*). In the following examples we fix an enumeration $x_1, \ldots, x_n$ of the variables; any other enumeration leads to a different ordering.

(1) Global orderings:

(i) *Lexicographical ordering* $>_{lp}$,

$$x^\alpha >_{lp} x^\beta :\Leftrightarrow \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$

(ii) *Degree lexicographical ordering* $>_{Dp}$,

$$x^\alpha >_{Dp} x^\beta :\Leftrightarrow \deg x^\alpha > \deg x^\beta$$
$$\text{or} \quad (\deg x^\alpha = \deg x^\beta \quad \text{and} \quad \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i).$$

(iii) *Weighted degree lexicographical ordering* $Wp(\omega_1, \ldots, \omega_n)$.

Given a vector $\omega = (\omega_1, \ldots, \omega_n)$ of integers, we define the *weighted degree* of x^α by $\deg_\omega(x^\alpha) := \langle \omega, \alpha \rangle := \omega_1\alpha_1 + \cdots + \omega_n\alpha_n$, that is, the variable x_i has degree ω_i. For a polynomial $f = \sum_\alpha a_\alpha x^\alpha$, we define the *weighted degree*,

$$\deg_\omega(f) := \max\{\deg_\omega(x^\alpha) \mid a_\alpha \neq 0\}.$$

Using the weighted degree in (ii), with all $\omega_i > 0$, instead of the usual degree, we obtain the weighted degree lexicographical ordering, $Wp(\omega_1, \ldots, \omega_n)$.

(2) Local orderings:

(i) *Negative lexicographical ordering* $>_{ls}$,

$$x^\alpha >_{ls} x^\beta :\Leftrightarrow \exists 1 \le i \le n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i.$$

(ii) *Negative degree lexicographical ordering* $>_{Ds}$,

$$x^\alpha >_{Ds} x^\beta :\Leftrightarrow \deg x^\alpha < \deg x^\beta$$
$$\text{or} \quad (\deg x^\alpha = \deg x^\beta \quad \text{and} \quad \exists 1 \le i \le n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i).$$

(iii) *Negative weighted degree lexicographical ordering* $Ws(\omega_1, \ldots, \omega_n)$; this is a weighted version of the last ordering.

2.3.2. Notation

Definition 2.12. Let $>$ be a fixed monomial ordering. Write $f \in K[x]$, $f \neq 0$, in a unique way as a sum of non-zero terms

$$f = a_\alpha x^\alpha + a_\beta x^\beta + \cdots + a_\gamma x^\gamma, \quad x^\alpha > x^\beta > \cdots > x^\gamma,$$

and $a_\alpha, a_\beta, \ldots, a_\gamma \in K$. We define:

(1) $LM(f) := x^\alpha$, the leading monomial of f,
(2) $LE(f) := \alpha$, the leading exponent of f,
(3) $LT(f) := a_\alpha x^\alpha$, the leading term of f,
(4) $LC(f) := a_\alpha$, the leading coefficient of f,
(5) $tail(f) := f - LT(f) = a_\beta x^\beta + \cdots + a_\gamma x^\gamma$, the tail of f.

Definition 2.13. For any monomial ordering $>$ on $\mathrm{Mon}(x_1, \ldots, x_n)$, we define the ring $K[x]_>$ associated with $K[x]$ and $>$ by

$$K[x]_> := \left\{ \frac{f}{u} \mid f, u \in K[x], LM(u) = 1 \right\}.$$

Note that $K[x]_> = K[x]$ if and only if $>$ is global and $K[x]_> = K[x]_{\langle x_1, \ldots, x_n \rangle}$ if and only if $>$ is local.

2.3.3. Normal form

Let $>$ be a monomial ordering and let $R = K[x_1, \ldots, x_n]_>$ (cf. Definition 2.13 above). For any subset $G \subset R$ define the ideal

$$L_>(G) := L(G) := \langle LM(g) | g \in G \setminus \{0\} \rangle_{K[x]}.$$

$L(G) \subset K[x]$ is called the *leading ideal* of G. Note that if I is an ideal, then $L(I)$ is the ideal generated by all leading monomials of all elements of I and not only by the leading monomials of a given set of generators of I.

Definition 2.14. Let $\mathcal{G}$ denote the set of all finite subsets $G \subset R$. A map

$$NF : R \times \mathcal{G} \to R, \quad (f, G) \mapsto NF(f \mid G),$$

is called a *normal form* on R if, for all $f \in R$ and $G \in \mathcal{G}$,

(0) $NF(0|G) = 0$;
(1) $NF(f|G) \neq 0 \Rightarrow LM(NF(f|G)) \notin L(G)$;
(2) if $G = \{g_1, \ldots, g_s\}$, then $r := f - NF(f|G)$ has a standard representation with respect to G, that is, either $r = 0$, or

$$r = \sum_{i=1}^{s} a_i g_i, \quad a_i \in R,$$

satisfying $LM(f) \ge LM(a_i g_i)$ for all i such that $a_i g_i \neq 0$.

NF is called a *reduced normal form* if, moreover, $NF(f|G)$ is reduced with respect to G, i.e. no monomial of the power series expansion of $NF(f|G)$ is contained in $L(G)$.

As we can see from the definition, $NF(f|G) = 0$ if and only if $f \in \langle G \rangle$.

2.3.4. RedNFBuchberger algorithm for computation of the normal form

Algorithm 2.15 (REDNFBUCHBERGER *Algorithm*). Assume that $>$ is a global monomial ordering.
Input: $f \in K[x]$, $G \in \mathcal{G}$.
Output: $p \in K[x]$, a reduced normal form of f with respect to G.

(1) $p := 0; h := f;$
(2) while $(h \neq 0)$
 (a) while $(h \neq 0$ and $G_h := \{g \in G \mid LM(g)$ divides $LM(h)\} \neq 0)$
  { choose any $g \in G_h$;
  $h := h - (LT(h)/LT(g)) \cdot g\}$
 (b) if $(h \neq 0)$
  $\{p := p + LT(h);$
  $h := tail(h)\};$
(3) return $p/LC(p)$.

The algorithm works in the following way. The inner loop (2a) runs until it meets an "obstruction", i.e. the first monomial that is not divisible by the leading monomial of any member of G. When the inner loop (2a) stops, h stores a normal form of f. To make this normal form reduced, we add the leading term of h, i.e. the "obstruction", to p and continue working with the tail of h in the same way.

Note that any specific choice of "any $g \in G_h$" can give a different normal form function. For the proof of correctness of the algorithm, cf. [5], Section 1.6, algorithms 1.6.10 and 1.6.11.

2.3.5. Highest corner

Definition 2.16. Let $>$ be a monomial ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$ and let $I \subset K[x_1, \ldots, x_n]_>$ be an ideal. A monomial $m \in \mathrm{Mon}(x_1, \ldots, x_n)$ is called the *highest corner* of I (with respect to $>$), denoted by $HC(I)$, if

(1) $m \notin L(I)$;
(2) $m' \in \mathrm{Mon}(x_1, \ldots, x_n), m' < m \Rightarrow m' \in L(I)$.

Lemma 2.17. *Let $>$ be a monomial ordering on* $\mathrm{Mon}(x_1, \ldots, x_n)$ *and let $I \subset K[x_1, \ldots, x_n]_>$ be an ideal. Let m be a monomial such that $m' < m$ implies $m' \in L(I)$. Let $f \in K[x_1, \ldots, x_n]$ such that $LM(f) < m$. Then $f \in I$.*

Lemma 2.18. *Let $>$ be a weighted degree ordering on* $\mathrm{Mon}(x_1, \ldots, x_n)$*. Moreover, let $f_1, \ldots, f_k$ be a set of generators of the ideal $I \subset K[x_1, \ldots, x_n]_>$ such that $J := \langle LM(f_1), \ldots, LM(f_k)\rangle$ has a highest corner $m := HC(J)$ and $f \in K[x_1, \ldots, x_n]_>$. If $LM(f) < HC(J)$ then $f \in I$.*

For proofs of these lemmas, cf. [5].

3. The proof of the main results

It is enough to prove Theorem 1.2 since Theorem 1.1 is clearly its special case.

3.1. The idea of the proof

We are interested only in the singular point $z = (1, 0, 0)$ of C. Therefore we pass to affine coordinates $x = \frac{x1}{x0}$, $y = \frac{x2}{x0}$. To verify T-smoothness we have to compute $h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d))$:

$$h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) = h^0(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) - h^0(\mathcal{O}_{P^2}(d)) + h^0(\mathcal{O}_{Z^{ea}(C,z)}).$$

The computations can be illustrated by means of a Newton diagram (cf. Fig. 1). $h^0(\mathcal{O}_{P^2}(d))$ equals the number of integer points in the triangle $\{(i, j) \mid i + j \leq d\}$. $h^0(\mathcal{O}_{Z^{ea}(C,z)})$ equals the number of integer points in the rectangle $\{(i, j) | i \leq k - 2, j \leq l - 2\}$. $h^0(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d))$ equals the number of integer points that lie inside the triangle but outside the rectangle. Hence $h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d))$ equals the number of integer points that lie inside the rectangle but outside the triangle, i.e. equals one (see the precise computation below in the proof of the theorem).

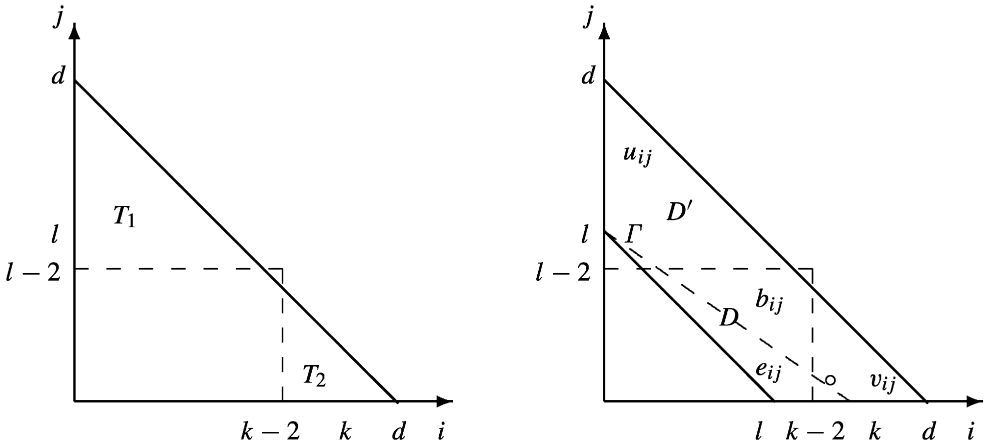

Fig. 1. Newton diagram.

For the same reason $h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d+1)) = 0$.

Therefore $V_{d+1,C}(S)$ is smooth, of expected codimension. Hence it is locally defined by τ analytic equations in coefficients with non-degenerate linearization.

We transform this system of equations to an equivalent system of equations with diagonal linear part consisting of the coefficients of the monomials from the rectangle.

Substituting zeros for coefficients of monomials of degree $d+1$ we obtain $\tau - 1$ equations with non-degenerate linear part and one more equation which starts from higher order terms. Then we prove that the principal part of the last equation is a non-degenerate quadratic form. For convenience we consider not the whole stratum, but a certain section of it.

3.2. *Proof of Theorem 1.2*

Proof. Let $f(x,y) = x^k + y^l$ where $k, l \geq 5$, $k \neq l$. Let C be the curve given by the equation $f(x,y) = 0$ and z be its singular point $(0,0)$. $f \in j(f) = \langle f_x, f_y \rangle = \langle x^{k-1}, y^{l-1} \rangle$; hence (C,z) is a quasihomogeneous plane curve singularity. The algebra $\mathbb{C}[x,y]/_{\langle f_x, f_y \rangle}$ has a basis $\{x^i y^j, 0 \leq i \leq k-2, 0 \leq j \leq l-2\}$. So $\tau(C,z) = (k-1)(l-1)$. The Newton diagram of (C,z) is $\Gamma : \frac{i}{k} + \frac{j}{l} = 1$.

Let $V_{d,C}(S)$ be the germ at C of the equisingular stratum $V_d(S)$ of all curves of degree d locally analytically equivalent to C. We can show that $V_{d,C}(S)$ is non-T-smooth at C and $h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) = 1$. Indeed $h^0(\mathcal{O}_{Z^{ea}(C,z)}) = \tau(C,z) = (k-1)(l-1)$,

$$H^0(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) = \left\{ \sum_{(i,j)\in T_1} a_{ij} x^i y^j + \sum_{(i,j)\in T_2} a_{ij} x^i y^j \right\}$$

where T_1 is the triangle $\{(i,j) | i+j \leq d, i \geq 0, j > l-2\}$, and T_2 is the triangle $\{(i,j) | i+j \leq d, j \geq 0, i > k-2\}$ (cf. Fig. 1). That means

$$h^0(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) = \frac{d(d+3)}{2} + 1 - ((k-1)(l-1) - 1).$$

From the exact sequence

$$0 \to H^0(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) \to H^0(\mathcal{O}_{P^2}(d)) \to H^0(\mathcal{O}_{Z^{ea}(C,z)}) \to H^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) \to 0.$$

We conclude that

$$\begin{aligned} h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) &= h^0(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d)) - h^0(\mathcal{O}_{P^2}(d)) + h^0(\mathcal{O}_{Z^{ea}(C,z)}) \\ &= \frac{d(d+3)}{2} + 2 - (k-1)(l-1) - \frac{d(d+3)}{2} - 1 + (k-1)(l-1) = 1. \end{aligned}$$

So by Proposition 2.8 our germ $V_{d,C}(S)$ is non-T-smooth. Now we would like to find out whether it is non-smooth or has wrong codimension. We will prove that it is non-smooth, of expected codimension.

For convenience we divide the proof to three cases: $l < k < 2l$, $k = l$ and $k \geq 2l$.

Case 1: $l < k < 2l$. First we will pass to a more convenient substratum, which has the same geometric properties. $V_d(S)$ is invariant under the action of PGL_3. Consider the subgroup of group PGL_3 generated by $x \mapsto x$, $y \mapsto y - ax$ and translations. We can choose a section of $V_d(S)$ transversal to orbits of this group and given by the conditions that the singularity is at the origin and the tangent to the curve at it is $\{y = 0\}$. The section is the substratum $V_d^l(S)$ of all curves C_F in $V_d(S)$ given by polynomials F which have zero coefficients of $x^i y^j$ for $i + j \leq l$ unless $j = l$. Note that fixing the singular point and the tangent has lowered the dimension by 3. In the same way, by considering the subgroup generated by $y \mapsto y$, $x \mapsto bx - cy$ we can reduce to substratum $V_d^{l,0,1}(S)$ of $V_d^l(S)$ consisting of all curves C_F given by polynomials F which also do not include the monomial $x^{k-1}y$ and have the same coefficients of x^k and y^l. Its dimension is $\dim V_d^{l,0,1}(S) = \dim V_d(S) - 5$.

So we will work with substratum germ $V_{d,C}^{l,0,1}(S)$ of $V_{d,C}(S)$ consisting of curves given by polynomials of the form

$$F(x, y) = y^l + x^k + \sum_{(i,j)\in\mathcal{D}} a_{ij} x^i y^j,$$
$$\text{where } \mathcal{D} = \{l < i + j \leq d, (i, j) \neq (k - 1, 1), (i, j) \neq (k, 0)\}.$$

For convenience we introduce the following notation:

(a) let e_{ij} be the coefficients of basis monomials which lie below Γ, i.e.

$$e_{ij} := a_{ij} \quad \text{for } \left\{(i, j) | 0 \leq i \leq k - 2, 0 \leq j \leq l - 2, \frac{i}{k} + \frac{j}{l} \leq 1\right\};$$

and let b_{ij} be the coefficients of basis monomials which lie above Γ, i.e.

$$b_{ij} := a_{ij} \quad \text{for } \left\{(i, j) | 0 \leq i \leq k - 2, 0 \leq j \leq l - 2, \frac{i}{k} + \frac{j}{l} > 1\right\};$$

(b) let $g_i := a_{i,l-1}$ and $h_j := a_{k-1,j}$, where $2 \leq i \leq d - (l - 1)$ and $2 \leq j \leq d - (k - 1)$;

(c) let $u_{ij} := a_{ij}$ for (i, j) from the triangle $\{i \geq 0, j \geq l, i + j \leq d\}$ and $v_{ij} := a_{ij}$ for (i, j) from the triangle $\{i \geq k, j \geq 0, i + j \leq d\}$.

Now we want to find out for which a_{ij} C_F lies in $V_{d,C}^{l,0,1}(S)$. Since $F(x, y)$ does not include the monomial xy^{l-1}, in order for F to have quasihomogeneous singularity of type $(1/k, 1/l, 1)$ all the coefficients of monomials under the Newton diagram Γ should vanish. In other words we have the following equations on a_{ij}:

$$e_{ij} = 0 \quad \text{for } \frac{i}{k} + \frac{j}{l} \leq 1, \text{ where } 0 \leq i \leq k - 2, 0 \leq j \leq l - 2 \tag{3.2.1}$$
$$h_0 := a_{k-1,0} = 0 \quad \text{if } k > l + 1. \tag{3.2.2}$$

So we continue to work with

$$F(x, y) = y^l + x^k + \sum_{(i,j)\in\mathcal{D}'} a_{ij} x^i y^j,$$
$$\text{where } \mathcal{D}' = \left\{(i, j) | i, j \geq 0, \frac{i}{k} + \frac{j}{l} > 1, i + j \leq d, (i, j) \neq (k - 1, 1)\right\}.$$

Since we require $z = (0, 0)$ to be a quasihomogeneous singularity, C_F lies in $V_{d,C}^{l,0,1}(S)$ if and only if $F(x, y) \in \langle F_x, F_y\rangle$ (cf. Section 2.1 and [7]).

In order to check whether $F(x, y) \in \langle F_x, F_y\rangle$ we use the REDNFBUCHBERGER algorithm (cf. Section 2.3.4). We refer to a small neighborhood of the origin; hence we consider $F(x, y)$ and $\langle F_x, F_y\rangle$ in the local ring $R = \mathbb{C}[x, y]_{\langle x,y\rangle}$. For computing in this ring we define a local monomial ordering on $\mathbb{C}[x, y]$ such that the ring associated with $\mathbb{C}[x, y]$ and this ordering will be $\mathbb{C}[x, y]_{\langle x,y\rangle}$. We choose the negative weighted degree lexicographical ordering with $w = (l, k)$ (cf. Example 2.11 (2)(iii)).

In general the REDNFBUCHBERGER algorithm does not stop for local orderings. However, in our case we can stop it manually when the leading monomial of the tail is less than $x^{k-2}y^{l-2}$. We are allowed to do that by Lemma 2.18, for $x^{k-2}y^{l-2}$ is the highest corner of $\langle LM(F_x), LM(F_y)\rangle = \langle x^{k-1}, y^{l-1}\rangle$. Indeed, any monomial smaller than $x^{k-2}y^{l-2}$ has either degree of x bigger than or equal to $k-1$, or degree of y bigger than or equal to $l-1$ and hence lies in $\langle LM(F_x), LM(F_y)\rangle$ and $x^{k-2}y^{l-2} \notin \langle LM(F_x), LM(F_y)\rangle$.

There is another explanation for why we can stop the algorithm at this point. Consider $V_{d+1,C}(S)$. As above, we compute $h^1(\mathcal{J}_{Z^{ea}(C,z)/\mathbb{P}^2}(d+1)) = 0$. Hence $V_{d+1,C}(S)$ is smooth, of expected codimension. So $V^{l,0,1}_{d+1,C}(S)$ is also smooth and has expected codimension which is equal to the number of basis elements which lie above the Newton diagram. Since we have exactly this number of independent equations at this stage, there will be no more equations. Also since $V^{l,0,1}_{d+1,C}(S)$ is smooth, all the equations on it will have independent linear parts. When we return to $V^{l,0,1}_{d,C}(S)$, the linear part of only one of them may vanish.

So we rewrite the algorithm in the following way:

Algorithm 3.1 (*Modified* REDNFBUCHBERGER *Algorithm*).

(1) $p := 0, h := F$;
(2) while ($h \neq 0$ and $LM(h) \geq x^{k-2}y^{l-2}$)
  (a) while ($h \neq 0$ and $LM(h) \geq x^{k-2}y^{l-2}$ and
      ($LM(F_x)$ divides $LM(h)$ or $LM(F_y)$ divides $LM(h)$))
    (i) if ($LM(F_x)$ divides $LM(h)$)
        $\{h := h - (LT(h)/LT(F_x)) \cdot F_x\}$
    (ii) if ($h \neq 0$ and $LM(F_y)$ divides $LM(h)$)
        $\{h := h - (LT(h)/LT(F_y)) \cdot F_y\}$
  (b) if ($h \neq 0$ and $LM(h) \geq x^{k-2}y^{l-2}$)
      $\{p := p + LT(h)$;
      $h = tail(h)\}$;
(3) return p.

As a result we obtain the normal form

$$NF(F|\langle F_x, F_y\rangle) = \sum R_{mn}(a_{ij})x^m y^n,$$

where $x^m y^n, 0 \leq m \leq k-2, 0 \leq n \leq l-2$ are elements of the basis of algebra $\mathbb{C}[x, y]/_{\langle f_x, f_y\rangle}$ which lie above Γ and $R_{mn}(a_{ij})$ are polynomials in a_{ij}. Hence F belongs to the ideal $\langle F_x, F_y\rangle$ if and only if all coefficients $R_{mn}(a_{ij}) = 0$.

Let $I = \langle g_2, \ldots, g_{d-(l-1)}, h_2, \ldots, h_{d-(k-1)}\rangle$ and $H = \langle b_{ij}\rangle$. Transforming our system of equations to an equivalent one we obtain

$$b_{ij} = \psi_{ij}(g_2, \ldots, g_{d-(l-1)}, h_2, \ldots, h_{d-(k-1)}, u_{pq}, v_{rs}, b_{mn}) \quad \text{for } (i, j) \neq (k-2, l-2) \quad \text{and}$$
$$0 = \psi_{k-2,l-2}(g_2, \ldots, g_{d-(l-1)}, h_2, \ldots, h_{d-(k-1)}, u_{pq}, v_{rs}, b_{mn}),$$

where ψ_{ij} is a polynomial from $I^2 + H$ and all its monomials have degree at least 2.

Note that this system is sorted by the quasihomogeneous degree of the left hand side and all b_{mn} in the right hand sides are left hand sides of previous equations. Hence we can replace our system by an equivalent one:

$$b_{ij} = \widetilde{\psi}_{ij}(g_2, \ldots, g_{d-(l-1)}, h_2, \ldots, h_{d-(k-1)}, u_{pq}, v_{rs}) \quad \text{for } (i, j) \neq (k-2, l-2) \tag{3.2.3}$$

where $\widetilde{\psi}_{ij} \in I^2$ are polynomials in $(g_i, h_j, u_{pq}, v_{rs})$ and

$$G := \sum_{i=2}^{d-(l-1)} \frac{l-1}{l} g_i g_{k-2-i} + \sum_{j=2}^{d-(k-1)} \frac{k-1}{k} h_j h_{l-2-j}$$
$$+ \Theta(g_2, \ldots, g_{d-(l-1)}, h_2, \ldots, h_{d-(k-1)}, u_{pq}, v_{rs}) = 0, \tag{3.2.4}$$

where $\Theta(g_2, \ldots, g_{d-l+1}, h_2, \ldots, h_{d-k+1}, u_{pq}, v_{rs})$ is a polynomial from I^2 and all its monomials are of degree at least 3.

Merging this system with (3.2.1) and (3.2.2) we see that our substratum germ is isomorphic to the germ at 0 of the affine variety given in the affine space with coordinates $\{g_2, \dots, g_{d-l+1}, h_2, \dots, h_{d-k+1}, u_{pq}, v_{rs}\}$ by the last Eq. (3.2.4).

Let us now prove that the same is true in the case $k = l$, and then finish the proof for both cases.

Case 2: $k = l$. As in the previous case using PGL_3 we reduce our consideration to the substratum $V_d^l(S)$ consisting of curves having singular point at the origin with the same tangent cone as C. By using the subgroup of $PGL_3 x \mapsto x/a$, $y \mapsto y$ we pass to the substratum $V_d^{l,1}(S)$ of curves given by equations $F(x, y) = 0$, where

$$F(x,y) = x^l + y^l + \sum_{l<i+j\leq d} a_{ij}x^i y^j.$$

Note that $\dim V_d^{l,1}(S) = \dim V_d(S) - 5$. Using the modified REDNFBUCHBERGER algorithm (Algorithm 3.1), we obtain a system of equations on $V_{d,C}^{l,1}(S)$. As in the previous case, we transform this system of equations to an equivalent system of equations with diagonal linear part. The first equations express all the basis elements above $i + j = l$ through other coefficients and the last one is (3.2.4).

So, as in the previous case, our substratum germ is isomorphic to the germ at 0 of the affine variety given in the affine space W with coordinates $\{g_2, \dots, g_{d-l+1}, h_2, \dots, h_{d-k+1}, u_{pq}, v_{rs}\}$ by the last Eq. (3.2.4).

(i) For $l = 5, k = 6, d = k + l - 5 = 6$ the points of $V_{6,C}^{5,0,1}(S)$ are curves given by equations of the form

$$F(x, y) = y^5 + x^6 + b_{42}x^4y^2 + b_{33}x^3y^3 + g_{24}x^2y^4 + u_{15}xy^5 + u_{06}y^6.$$

Using the modified REDNFBUCHBERGER algorithm (Algorithm 3.1) we obtain $b_{42} = 0, b_{33} = 0$, and $(g_{24})^2 = 0$, which implies the statement (i).

(ii) For $l = 5, k = 7, d = k + l - 5 = 7$ the variety $V_{7,C}^{5,0,1}(S)$ is given by the equation

$$g_{24}\left(g_{34} - \frac{1}{2}g_{24}u_{15}\right) = 0.$$

So it is reducible and decomposes into two smooth components of expected codimension that intersect transversally in W. Hence, the two components of $V_{7,C}(S)$ intersect in $V_{|C|}$ non-transversally with multiplicity one and the sectional singularity is of type A_1.

For $k = l = 6, d = k + l - 5 = 7$ the variety $V_{7,C}^{6,1}(S)$ is given by the equation $h_{52}^2 + g_{25}^2 = 0$. So it is reducible and decomposes into two smooth components of expected codimension that intersect transversally in W. Hence, the two components of $V_{7,C}(S)$ intersect in $V_{|C|}$ non-transversally with multiplicity one and the sectional singularity is of type A_1.

(iii) In this case $k, l \geq 5, d = k + l - 5 \geq 8$. The quadratic part of Eq. (3.2.4) is

$$Q := \sum_{i=2}^{d-(l-1)} A_i g_i g_{d-l+3-i} + \sum_{j=2}^{d-(k-1)} B_j h_j h_{d-k+3-j}.$$

Since A_i and B_i are non-zero, it is a quadratic non-degenerate form of rank $r = d - l + d - k = 2d - (l + k) = d - 5$. Since for $d \geq 8$ the rank $r = d - 5 \geq 3$, our substratum germ is a reduced irreducible non-smooth variety of expected codimension.

Now we are going to prove that the singular locus Y of our substratum germ coincides with the germ X_0 at the origin of the affine subspace $X = Z(I)$. Since Eq. (3.2.4) lies in I^2, all its first-order partial derivatives lie in I, and hence the singular locus includes X_0. Let Z be the variety given by the equations

$$\frac{\partial G}{\partial g_2} = 0, \dots, \frac{\partial G}{\partial g_{d-l+1}} = 0, \qquad \frac{\partial G}{\partial h_2} = 0, \dots, \frac{\partial G}{\partial h_{d-k+1}} = 0.$$

Since the linear part of this system of equations is non-degenerate, the germ Z_0 of Z at the origin is smooth and hence irreducible. Clearly $Y \subset Z_0$ and hence $X_0 \subset Z_0$. They have the same dimension and Z_0 is irreducible; hence $X_0 = Z_0$ which implies $X_0 = Y$.

So our substratum germ is a reduced irreducible non-smooth variety of expected codimension which has smooth singular locus with sectional singularity of type A_1.

Case 3: $k \geq 2l$. In the same way as in case 1 we will pass to more convenient substratum, which has the same geometric properties. First we pass to the substratum $V_d^l(S)$ of all curves C_F in $V_d(S)$ given by polynomials F which have zero coefficients of $x^i y^j$ for $i + j \leq l$ unless $j = l$. Then by considering the subgroup generated by $y \mapsto y, x \mapsto x - by$ we can reduce to the substratum $V_d^{l,0}(S)$ of $V_d^l(S)$ consisting of all curves C_F in $V_d^l(S)$ given by polynomials F which also do not include the monomial $x^{k-1}y$. Note that $\dim V_d^{l,0}(S) = \dim V_d(S) - 4$.

So we will work with the substratum germ $V_{d,C}^{l,0}(S)$ of $V_{d,C}(S)$ consisting of curves given by polynomials of the form

$$F(x, y) = y^l + x^k + \sum_{(i,j)\in\mathcal{D}} a_{ij} x^i y^j, \quad \text{where } \mathcal{D} = \{l < i + j \leq d, (i, j) \neq (k - 1, 1)\}.$$

Now we make the following series of $[k/l]-1$ coordinate changes. The first one will be $x \mapsto x$, $y \mapsto y - \frac{g_2}{l}x^2$. The coefficients of the polynomial F in the new coordinates are expressed through the coefficients in the old coordinates by the formula

$$a_{ij}^{(2)} = a_{ij} + \sum_{n=1}^{[i/2]} (-1)^n C_{j+n}^n a_{i-2n,j+n} \left(\frac{g_2}{l}\right)^n,$$

We continue in the same way. The coordinate change number $s - 1$ will be $x \mapsto x$, $y \mapsto y - \frac{g_s}{l}x^s$ and

$$a_{ij}^{(s)} = a_{ij}^{(s-1)} + \sum_{n=1}^{[i/s]} (-1)^n C_{j+n}^n a_{i-sn,j+n}^{(s-1)} \left(\frac{g_s^{(s-1)}}{l}\right)^n. \tag{3.2.5}$$

We stop when $s = [k/l]$. Clearly the degrees of polynomials from $\mathcal{P}(d)$ remain universally bounded after these coordinate changes. Denote this bound by N.

Let a'_{ij} be the coefficients of our polynomials after all these coordinate changes, i.e. $a'_{ij} := a_{ij}^{([k/l])}$. Clearly $g'_i = 0$, $i = 2, \ldots, [k/l]$. Since the coordinate changes (3.2.5) were local analytic diffeomorphisms, the curve C_F given by the polynomial equation $F(x, y) = 0$ lies in $V_{d,C}^{l,0}(S)$ if and only if the curve $C_{\widetilde{F}}$ given by $\widetilde{F}(x, y) = 0$ lies in $V_{N,C}^{l,0}(S)$ where

$$\widetilde{F}(x, y) = y^l + x^k + \sum a'_{ij} x^i y^j.$$

Now we want to find out for which a'_{ij} $C_{\widetilde{F}}$ lies in $V_{N,C}^{l,0}(S)$. Since $\widetilde{F}(x, y)$ does not include the monomials $x^i y^{l-1}$ for $i = 1, \ldots, [k/l]$, in order for $\widetilde{F}$ to have quasihomogeneous singularity of type $(1/k, 1/l, 1)$ all the coefficients of monomials under the Newton diagram Γ should vanish. In other words we have the following equations on a'_{ij}:

$$e'_{ij} = 0 \quad \text{for } \frac{i}{k} + \frac{j}{l} \leq 1, \text{ where } 0 \leq i \leq k - 2, 0 \leq j \leq l - 2 \tag{3.2.6}$$

$$h'_0 := a'_{k-1,0} = 0 \tag{3.2.7}$$

So we continue to work with

$$\widetilde{F}(x, y) = y^l + (1 + a'_{k0})x^k + \sum_{(i,j)\in\mathcal{D}''} a'_{ij} x^i y^j, \quad \text{where } \mathcal{D}'' = \left\{(i, j) | i, j \geq 0, \frac{i}{k} + \frac{j}{l} > 1, i + j \leq N\right\}.$$

Since we require $z = (0, 0)$ to be a quasihomogeneous singularity, $C_{\widetilde{F}}$ lies in $V_{N,C}^{l,0}(S)$ if and only if $\widetilde{F}(x, y) \in \langle \widetilde{F}_x, \widetilde{F}_y \rangle$ (cf. Section 2.1 and [7]).

As in the previous cases, in order to check whether $\widetilde{F}(x, y) \in \langle \widetilde{F}_x, \widetilde{F}_y \rangle$ we use the modified REDNFBUCHBERGER algorithm (Algorithm 3.1).

As a result we obtain the normal form

$$NF(\widetilde{F} | \langle \widetilde{F}_x, \widetilde{F}_y \rangle) = \sum R_{mn}(a'_{ij}) x^m y^n,$$

where $x^m y^n$, $0 \leq m \leq k - 2$, $0 \leq n \leq l - 2$, are elements of the basis of algebra $\mathbb{C}[x, y] /_{\langle f_x, f_y \rangle}$ which lie above Γ and $R_{mn}(a'_{ij})$ are rational functions in a'_{ij} whose denominators are polynomials in a'_{k0} non-vanishing at zero. Hence $\widetilde{F}$ belongs to the ideal $\langle \widetilde{F}_x, \widetilde{F}_y \rangle$ if and only if all coefficients $R_{mn}(a'_{ij}) = 0$.

As in previous cases we transform this system of equations to an equivalent one and obtain

$$b'_{ij} = \frac{\psi'_{ij}(g'_{[k/l]+1}, \ldots, g'_{l-3-[k/l]}, h'_{[l/k]+1}, \ldots, h'_{k-3-[l/k]}, u'_{pq}, v'_{rs})}{(1+v'_{k0})^{\gamma_{i,j}}} \tag{3.2.8}$$

where $\psi'_{ij} \in I'^2$ for $I' = \langle g'_{[k/l]+1}, \ldots, g'_{l-3-[k/l]}, h'_{[l/k]+1}, \ldots, h'_{k-3-[l/k]} \rangle$.

The number of equations in system (3.2.8) is equal to the number of basis coefficients above the Newton diagram Γ.

Now we express new coefficients through the old ones. As we can see from (3.2.5), $a'_{ij} = a_{ij} + \varphi_{ij}(a_{pq})$ for $(i,j) \in \mathcal{D}$ and $a'_{ij} = \varphi_{ij}(a_{pq})$ for $(i,j) \notin \mathcal{D}$, where $\varphi_{ij}(a_{pq})$ is a polynomial in a_{pq} which lies in the ideal $I = \langle g_2, \ldots, g_{d-l+1}, h_2, \ldots, h_{d-k+1} \rangle$ and does not conclude linear terms.

Consider system (3.2.6). It is sorted by the quasihomogeneous degree of the left hand side. After substituting for old coefficients it will be $e_{ij} = -\varphi_{ij}(g_2, \ldots, g_{[k/l]}, e_{mn})$, where all e_{mn} are left hand sides of the previous equations. Hence we can replace our system by an equivalent one:

$$e_{ij} = -\widetilde{\varphi}_{ij}(g_2, \ldots, g_{[k/l]}) \quad \text{for all } (i,j).$$

Now we substitute $-\widetilde{\varphi}_{ij}$ in place of e_{ij} in all of the other equations. The additional Eq. (3.2.7) acquires the form $h_0 = \widetilde{\varphi}_{k-1,0}(g_2, \ldots, g_{[k/l]})$. Consider system (3.2.8) except the last equation, i.e. the equation on $b'_{k-2,l-2}$. After substituting for the old coefficients and multiplying by denominators it will become

$$b_{ij} = \widetilde{\psi}_{ij}(g_2, \ldots, g_{d-l+1}, h_2, \ldots, h_{d-k+1}, u_{pq}, v_{rs}, b_{mn}) \tag{3.2.9}$$

where $\widetilde{\psi}$ does not have either free or linear terms.

The last equation is of particular interest. After passing to the old coordinates its right hand side will be

$$\begin{aligned} &\sum_{i=[k/l]+1}^{d-(l-1)+2-([k/l]+1)} A_i g_i g_{d-(l-1)+2-i} + \sum_{j=2}^{d-(k-1)} B_j h_j h_{d-(k-1)+2-j} \\ &\quad + \widetilde{\Psi}(g_2, \ldots, g_{d-l+1}, h_2, \ldots, h_{d-k+1}, u_{pq}, v_{rs}, b_{mn}), \end{aligned} \tag{3.2.10}$$

where $\widetilde{\Psi}$ is a rational function regular at the origin whose numerator lies in I^2. The left hand side will be $0 + \sum_{i=2}^{[k/l]} A_i g_i g_{d-l+3-i} + \widetilde{\Phi}(g_2, \ldots, g_{d-l+1}, u_{pq})$, where $\widetilde{\Phi}(g_2, \ldots, g_{d-l+1}, u_{pq})$ is a polynomial from I^2, all its monomials are of degree at least 3 and A_i are non-zero real numbers. Multiplying by the denominator of $\widetilde{\Psi}$ we obtain the equation

$$\begin{aligned} G &:= \sum_{i=2}^{d-(l-1)} A_i g_i g_{d-l+3-i} + \sum_{j=2}^{d-(k-1)} B_j h_j h_{d-k+3-j} \\ &\quad + \Theta(g_2, \ldots, g_{d-l+1}, h_2, \ldots, h_{d-k+1}, u_{pq}, v_{rs}) = 0, \end{aligned} \tag{3.2.11}$$

where $\Theta(g_2, \ldots, g_{d-l+1}, h_2, \ldots, h_{d-k+1}, u_{pq}, v_{rs})$ lies in I^2, all its monomials are of degree at least 3 and A_i, B_j are non-zero real numbers.

So our substratum germ $V^{l,0}_{d,C}(S)$ is isomorphic to the germ at 0 of the affine variety given in the affine space with coordinates $\{g_2, \ldots, g_{d-l+1}, h_2, \ldots, h_{d-k+1}, u_{pq}, v_{rs}, b_{ij}\}$ by the system of equations (3.2.9) with diagonal linear part and the last equation (3.2.11).

Since in this case $d = k + l - 5 \geq 3l - 5 \geq 10$, we have to prove that $V^{l,0}_{d,C}(S)$ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus with sectional singularity of type A_1.

The system (3.2.9) has diagonal linear part. Hence all the equations in it are independent and the variety W defined by it is smooth at 0. The last equation (3.2.11) does not depend on the preceding ones and does not have a linear part. Hence $V^{l,0}_{d,C}(S)$ is non-smooth, of expected codimension.

The quadratic part of Eq. (3.2.11) is

$$Q := \sum_{i=2}^{d-(l-1)} A_i g_i g_{d-l+3-i} + \sum_{j=2}^{d-(k-1)} B_j h_j h_{d-k+3-j}.$$

Since A_i, B_j are non-zero, it is a quadratic non-degenerate form of rank

$$r = d - l + d - k = 2d - (l + k) = d - 5 \geq 5. \tag{3.2.12}$$

Hence the variety $\{G = 0\}$ is reduced and irreducible and hence our germ is also reduced and irreducible.

Now we are going to prove that the singular locus Y of our substratum germ coincides with the germ X_0 at the origin of the affine subvariety X given in W by the ideal I.

Since Eq. (3.2.11) lies in I^2, all of its first-order partial derivatives lie in I, and hence X_0 lies in the singular locus. Consider the jacobian of the system obtained by merging system (3.2.9) with the last equation (3.2.11). Let $M(i)$ be its minor given by columns that include partial derivatives with respect to b_{mn}, and g_i and $N(j)$ be its minor given by columns that include partial derivatives with respect to b_{mn} and h_j. The linear part of $M(i)$ is $A_i g_{d-l-3-i}$ and that of $N(j)$ is $B_j g_{d-k-3-j}$. Let Z_0 be the germ at the origin of the subvariety of W given by the system of equations $M(i) = 0$ and $N(j) = 0$. Since the linear part of this system is diagonal, Z_0 is irreducible and has the same dimension as X_0. As $X_0 \subset Y \subset Z_0$, this implies $X_0 = Y = Z_0$.

So the substratum germ $V^{l,0}_{d,C}(S)$ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus with sectional singularity of type A_1. $\square$

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